Coordination Failure in Repeated Games with Almost-Public Monitoring

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Abstract

We describe the sense in which some private-monitoring games, that is, games with no public histories, can have histories that are almost public. These games are the natural result of perturbing public-monitoring games towards private monitoring. We explore the extent to which it is possible to coordinate continuation play in such games. It is always possible to coordinate continuation play by requiring behavior to have bounded recall (i.e., there is a bound \( L \) such that in any period, the last \( L \) signals are sufficient to determine behavior). We show that, in games with general almost-public private monitoring, this is essentially the only behavior that can coordinate continuation play.

Keywords: repeated games, private monitoring, almost-public monitoring, coordination, bounded recall.

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Coordination Failure in Repeated Games with Almost-Public Monitoring

by George J. Mailath and Stephen Morris

1. Introduction

Intertemporal incentives often allow players to achieve payoffs that are inconsistent with myopic incentives. For games with public histories (monitoring), the construction of sequentially rational equilibria with nontrivial intertemporal incentives is straightforward. Since continuation play in a public strategy profile is a function of public histories only, the requirement that continuation play induced by any public history constitute a Nash equilibrium of the original game is both the natural notion of sequential rationality and relatively easy to check (Abreu, Pearce, and Stacchetti (1990)). These perfect public equilibria (or PPE) use public histories to coordinate continuation play.

While games with private monitoring (where actions and signals are private) have no public histories to coordinate continuation play, some do have histories that are almost public. We explore the extent to which it is possible to coordinate continuation play for such games. It is always possible to coordinate continuation play by requiring behavior to have bounded recall (i.e., there is a bound $L$ such that in any period, the last $L$ signals are sufficient to determine behavior). We show that, in games with general almost-public private monitoring, this is essentially the only behavior that can coordinate continuation play. To make this precise, we must describe what it means for a game to have “general but almost-public private monitoring” and “essentially.”

Since the coordination-of-continuation-play interpretation depends on the structure of the strategy profile, we focus on equilibrium strategy profiles, rather than on the equilibrium payoff set, of private-monitoring games. Very little is known about the general structure of the equilibrium payoff set for general private-monitoring games. We return to this issue at the end of the Introduction.

Fix a game with full support public monitoring (so that every signal arises with strictly positive probability under every action profile). In the minimal perturbation of the public-monitoring game towards private monitoring, each player observes a private signal drawn from the space of public signals, and the other specifications of the game are unchanged. In this private-monitoring game, at the end of each period, there is a profile of private signals, and we say the game has minimally-private almost-public monitoring if the probability of any profile in which all players observe the same value of the signal is close to the probability of that signal in the public-monitoring game (there is also positive probability that different players observe different values of the public signal).
Any strategy profile of a public-monitoring game naturally induces behavior in minimally-private almost-public-monitoring games.\footnote{Since player i’s set of histories in the public-monitoring game and in the minimally-private almost-public-monitoring game agree, the domains for player i’s strategy in the two games also agree.} Mailath and Morris (2002) introduced a useful representation device for these profiles. Recall that all PPE of a public-monitoring game can be represented in a recursive way by specifying a state space, a transition function mapping public signals and states into new states, and decision rules for the players, specifying behavior in each state (Abreu, Pearce, and Stacchetti (1990)). We use the same state space, transition function and decision rules to summarize behavior in the private-monitoring game. Each player will now have a private state, and the transition function and decision rules define a Markov process on vectors of private states.

This representation is sufficient to describe behavior under the given strategies, but (with private monitoring) is not sufficient to verify that the strategies are optimal. It is also necessary to know how each player’s beliefs over the private states of other players evolve. This is at the heart of whether histories can coordinate continuation play, since, given a strategy profile, a player’s private state determines that player’s continuation play. A sufficient condition for a strict equilibrium to remain an equilibrium with private monitoring is that after every history each player assigns probability uniformly close to one to all other players being in the same private state (Mailath and Morris (2002, Theorem 4.1)). PPE with bounded recall satisfy this sufficient condition, since for sufficiently close-by games with minimally-private almost-public monitoring, the probability that all players observed the same last $L$ signals can be made arbitrarily close to one. However, under other strategy profiles, the condition may fail. The grim trigger PPE in some parameterizations of the repeated prisoners’ dilemma, for example, does not induce an equilibrium in any close-by minimally-private almost-public monitoring game (Example 2 in Section 3.1).

The restriction to minimally-private almost-public monitoring is substantive, since all players’ private signals are drawn from a common signal space. In this paper, consistent with the game being “close-to” a public-monitoring game, we allow for significantly more general private monitoring. We assume there is a signalling function for each player that assigns to each private signal either some value of the public signal or a dummy signal (with the interpretation that that private signal cannot be related to any public signal). Using these signalling functions (one for each player), there is a natural sense in which the private monitoring distribution can be said to be close to the public monitoring distribution, even when the sets of private signals differ, and may have significantly larger cardinality than that of the set of public signals. We say such games have almost-public monitoring. If every private signal is mapped to a public signal, we say the almost-public monitoring game is strongly close to the public-monitoring game.
Using the signalling functions, any strategy profile of the public-monitoring game induces behavior in strongly-close-by almost-public monitoring games. As in minimally-private almost-public-monitoring games, a player’s private state determines that player’s continuation play. Given a sequence of private signals for a player, that player’s private state is determined by the induced sequence of public signals that are the result of applying his signalling function. Consequently, it might appear that the richness of the private signals does not alter the situation from the case of minimally-private almost-public monitoring. However, the richness of the private signals is important for the formation of that player’s beliefs about the other players’ private states. It turns out that the requirement that the private-monitoring distribution be close to the public-monitoring distribution places essentially no restriction on the manner in which private signals enter into the formation of posterior beliefs. Nonetheless, if the profile has bounded recall, the richness of the private signals is irrelevant. Indeed, even if the private-monitoring games are not strongly close to the public-monitoring game, there is still a natural sense in which every strict PPE with bounded recall induces equilibrium behavior in every close-by almost-public-monitoring game (Theorem 2).

When a strategy profile of the public-monitoring game does not have bounded recall, realizations of the signal in early periods can have long-run implications for behavior. Subject to some technical caveats, we call such a profile separating. While the properties of bounded recall and separation do not exhaust possible behavior, they do appear to cover essentially all behaviors of interest. When the space of private signals is sufficiently rich in the values of posterior-odds ratios (this is what we mean by “general almost public”), and the profile is separating, it is possible to manipulate a player’s posterior over other players’ private states through an appropriate choice of private history. In particular, it is possible to choose a private history so that a player is in one private state and assigns arbitrarily high probability to all the other players being in a different common private state. At such a posterior, it is no longer optimal for the player to follow the profile at his private state. This gives us our main result (Theorem 4): Separating strict PPE profiles of public-monitoring games do not induce Nash equilibria in any strongly-close-by games with rich private monitoring.

Thus, separating strict PPE of public-monitoring games are not robust to the introduction of even a minimal amount of private monitoring. Consequently, separating behavior in private-monitoring games typically cannot coordinate continuation play (Corollary 1). On the other hand, bounded recall profiles are robust to the introduction of private monitoring. The extent to which bounded recall is a substantive restriction on the set of payoffs is unknown.\(^2\) Our results do suggest, even for public-monitoring games, bounded recall profiles are particularly attractive (since they are robust to the

\(^2\)Cole and Kocherlakota (2003) show that for some parameterizations of the repeated prisoners’ dilemma, the restriction to strongly symmetric bounded recall PPE results in a dramatic collapse of the set of equilibrium payoffs.
introduction of private monitoring). Moreover, other apparently simple strategy profiles are problematic.

Our focus on equilibrium strategy profiles is in contrast with much of the literature in repeated games with private monitoring. For the repeated prisoners’ dilemma with almost-perfect private monitoring, folk theorems have been proved using both equilibria with a coordination interpretation (for example, Sekiguchi (1997), which we discuss in Example 1, and Bhaskar and Obara (2002)) and those that are “belief-free” (for example, Piccione (2002), Ely and Välimäki (2002), and Matsushima (2004)). Loosely, belief-free equilibria are constructed so that after relevant histories, players are indifferent between different choices. In games with finite signal spaces, this requires a significant amount of randomization (randomization is not required with a continuum of signals, but only because behavior can be purified using signals). Not only is the generality of this approach unclear (Ely, Hörner, and Olszewski (2003)), the equilibria do not have a clean coordination interpretation. For example, the profiles in Ely and Välimäki (2002) are obtained from similar profiles in the repeated prisoners’ dilemma with perfect monitoring. The perfect-monitoring profiles also have a significant amount of randomization and are difficult to purify in the sense of Harsanyi (1973). More specifically, while those profiles only depend on the previous period’s actions, in the relevant equilibrium of the purifying game, behavior typically has unbounded memory (Bhaskar, Mailath, and Morris (2004)).

Finally, we view our findings as underlining the importance of communication in private-monitoring games as a mechanism to facilitate coordination. For some recent work on communication in private-monitoring games, see Compte (1998), Kandori and Matsushima (1998), Fudenberg and Levine (2004), and McLean, Obara, and Postlewaite (2002).

2. Private-Monitoring Games

The infinitely-repeated game with private monitoring is the infinite repetition of a stage game in which at the end of the period, each player learns only the realized value of a private signal. There are $n$ players, with a finite stage-game action set for player $i \in N \equiv \{1, \ldots, n\}$ denoted $A_i$. At the end of each period, each player $i$ observes a private signal, denoted $\omega_i$ drawn from a finite set $\Omega_i$. The signal vector $\omega \equiv (\omega_1, \ldots, \omega_n) \in \Omega \equiv \Omega_1 \times \cdots \times \Omega_n$ occurs with probability $\pi(\omega|a)$ when the action profile $a \in A \equiv \prod_i A_i$ is chosen. Player $i$ does not receive any information other than $\omega_i$ about the behavior of

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3 See Kandori (2002) for a brief survey of this literature, as well as the accompanying symposium issue of the Journal of Economic Theory on “Repeated Games with Private Monitoring.”

4 Matsushima (2004) covers general two player games with private monitoring that need not be almost perfect, with signals that are either conditionally independent or have a particular correlation structure. His analysis does not cover almost-public monitoring games.
the other players. All players use the same discount factor, $\delta$.

Since $\omega_i$ is the only signal a player observes about opponents’ play, we assume (as usual) that player $i$’s payoff after the realization $(\omega, a)$ is given by $u^*_i (\omega_i, a_i)$. Stage game payoffs are then given by $u_i (a) \equiv \sum_{\omega} u^*_i (\omega_i, a_i) \pi (\omega | a)$.

It will be convenient to index games by the monitoring technology $(\Omega, \pi)$, fixing the set of players and action sets.

A pure strategy for player $i$ in the private-monitoring game is a function $s_i : \mathcal{H}_i \rightarrow A_i$, where

$$\mathcal{H}_i \equiv \cup_{t=1}^{\infty} (A_i \times \Omega_i)^{t-1}$$

is the set of private histories for player $i$.

**Definition 1** A pure strategy is action-free if, for all $h^t_i, \hat{h}^t_i \in \mathcal{H}_i$ satisfying $\omega^\tau_i = \hat{\omega}^\tau_i$ for all $\tau \leq t$,

$$s_i(h^t_i) = s_i(\hat{h}^t_i).$$

Since action-free strategies play a central role in our analysis, it is useful to note the following immediate result (which does not require full-support monitoring):

**Lemma 1** Every pure strategy in a private-monitoring game is realization equivalent to an action-free strategy. Every mixed strategy is realization equivalent to a mixture over action-free strategies.

**Remark 1** The behavior strategy that is realization equivalent to a mixed strategy will typically not be action-free. Consider, as an example, the once repeated prisoners’ dilemma, with $\Omega_1 = \{g, b\}$ and the mixed strategy that assigns equal probability to the two action-free strategies $\bar{s}_1$ and $\tilde{s}_1$, where

$$\bar{s}_1 (\emptyset) = C; \bar{s}_1 (g) = C; \bar{s}_1 (b) = D,$$

and

$$\tilde{s}_1 (\emptyset) = D; \tilde{s}_1 (g) = D; \tilde{s}_1 (b) = D.$$

The behavior strategy that is realization equivalent to this mixed strategy must specify in the second period behavior that depends nontrivially on player 1’s first period action.

(A similar observation applies to public-monitoring games: every pure strategy is realization equivalent to a public strategy, every mixed strategy is realization equivalent to a mixture over public strategies, and the behavior strategy that is realization equivalent to a mixed strategy may not be public.)

Every pure action-free strategy can be represented by a set of states $W_i$, an initial state $w^1_i$, a decision rule $d_i : W_i \rightarrow A_i$ specifying an action choice for each state, and a transition function $\sigma_i : W_i \times \Omega_i \rightarrow W_i$. In the first period, player $i$ chooses action
\( a_1^i = d_i(w_1^i) \). At the end of the first period, the vector of actions, \( a^1 \), then generates a vector of private signals \( \omega^1 \) according to the distribution \( \pi(\cdot | a^1) \), and player \( i \) observes the signal \( \omega^1_i \). In the second period, player \( i \) chooses the action \( a_2^i = d_i(w_2^i) \), where \( w_2^i = \sigma_i(w_1^i, \omega_1^i) \), and so on. Any action-free strategy requires at most the countable set \( W_i = \bigcup_{t=1}^{\infty} \Omega_{i}^{t-1} \).

Any collection of pure action-free strategies can be represented by a set of states \( W_i \), a decision rule \( d_i \), and a transition function \( \sigma_i \) (the initial state indexes the pure strategies). One class of mixed strategies is described by \((W_i, \mu_i, d_i, \sigma_i)\), where \( \mu_i \) is a probability distribution over the initial state \( w_1^i \), and \( W_i \) is countable. Not all mixed strategies can be described in this way, since the set of all pure strategies is uncountable (which would require \( W_i \) to be uncountable).

**Remark 2** A consequence of Remark 1 is that typically, action-free strategy profiles, and profiles of mixtures over action-free strategies, cannot be sequentially rational. However, when the monitoring has full support, every Nash equilibrium has a realization-equivalent sequentially rational strategy profile (see Sekiguchi (1997, Proposition 3) and Kandori and Matsushima (1998, p. 648)). Consequently, we focus on Nash equilibria of games with private monitoring.

The repeated prisoners’ dilemma provides an informative example.

**Example 1** The ex ante stage game is given by

\[
\begin{array}{cc|cc}
| & | & e_1 & e_2 \\
|---|---|---|---|
| n_1 | 2, 2 & -1, 3 \\
| n_2 | 3, -1 & 0, 0 \\
\end{array}
\]

(1)

Much of the literature has studied almost-perfect conditionally-independent private monitoring: player \( i \)'s signals are given by \( \Omega_i = \{ \hat{e}_i, \hat{n}_i \} \), with \( \hat{a}_i \in \Omega_i \) a signal of \( a_j \in A_j \equiv \{ e_j, n_j \} \). Players 1 and 2’s signal are, conditional on the action profile, independently distributed, with

\[
\pi(\hat{a}_1 \hat{a}_2 | a_1 a_2) = \pi_1(\hat{a}_1 | a_2) \pi_2(\hat{a}_2 | a_1)
\]

and

\[
\pi_i(\hat{a}_i | a_j) = \begin{cases} 
1 - \varepsilon, & \text{if } \hat{a}_i = a_j, \\
\varepsilon, & \text{if } \hat{a}_i \neq a_j,
\end{cases}
\]

where \( \varepsilon > 0 \) is a small constant. As will be clear, we focus on a different class of private monitoring distributions.

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\(^5\)Here (and in other examples) we follow the literature in assuming the ex ante payoff matrix is independent of the monitoring distribution. This simplifies the discussion and is without loss of generality: Ex ante payoffs are close when the monitoring distributions are close (Lemma 4) and all relevant incentive constraints are strict.
Figure 1: The automaton described the pure strategies in Sekiguchi (1997) for $M = 3$. The decision rules are $d_i(w^{abc}) = a_i$. Unlabeled arrows are unconditional transitions.

In an important article, Sekiguchi (1997) constructed an efficient equilibrium for the almost-perfect conditionally-independent case (as well as for correlated but almost-perfect monitoring). Let $W_i = \{w^e, w^n\}$, $\sigma_i(w^n, \hat{a}_i) = w^n$ for all $\hat{a}_i$, $\sigma_i(w^e, \hat{e}_i) = w^e$, and $\sigma_i(w^e, \hat{n}_i) = w^n$. The pure strategy of grim trigger (begin playing $e_i$, and continue to play $e_i$ as long as $\hat{e}_i$ is observed, switch to $n_i$ after $\hat{n}_i$ and always play $n_i$ thereafter) is induced by the initial state $w_1^e = w^e$. The pure strategy of always play $n_i$ is induced by the initial state $w_1^n = w^n$. The critical insight in Sekiguchi (1997) is that while grim trigger is not a Nash equilibrium of this game, the symmetric mixed strategy profile where each player independently randomizes over initial states $w^e$ and $w^n$ is an equilibrium (as long as $\delta$ is not too close to 1). Sekiguchi (1997) then constructs an equilibrium for larger $\delta$ by treating the game as $M$ distinct games, with the $k$th game played in periods $t + kM$, for $t \in \mathbb{N}$. The mixed equilibrium for $M = 3$ is constructed from the machine in Figure 1. The state $w^{ene}$ for example corresponds to grim trigger in “games” 1 and 3, and always $n_i$ in game 2.

2.1. Public-Monitoring Games

We turn now to the benchmark public-monitoring game for our games with private monitoring. The finite action set for player $i \in N$ is again $A_i$. The public signal is denoted $y$ and is drawn from a finite set $Y$. The probability that the signal $y$ occurs when the action profile $a \in A = \prod_i A_i$ is chosen is denoted $\rho(y|a)$. We refer to $(Y, \rho)$ as
the public-monitoring distribution. Player $i$’s payoff after the realization $(y,a)$ is given by $\tilde{u}_i^*(y,a_i)$. Stage game payoffs are then given by $\tilde{u}_i(a) \equiv \sum_y \tilde{u}_i^*(y,a) \rho(y|a)$. The infinitely repeated game with public monitoring is the infinite repetition of this stage game in which at the end of the period each player learns only the realized value of the signal $y$. Players do not receive any other information about the behavior of the other players. All players use the same discount factor, $\delta$.

A strategy for player $i$ is public if, in every period $t$, it only depends on the public history $h^t \in \mathcal{Y}^{t-1}$, and not on $i$’s private history. Henceforth, by the term public profile, we will always mean a strategy profile for the public-monitoring game that is itself public. A perfect public equilibrium (PPE) is a profile of public strategies that, after observing any public history $h^t$, specifies a Nash equilibrium for the repeated game. Under imperfect full-support public monitoring, every public history arises with positive probability, and so every Nash equilibrium in public strategies is a PPE.

Any pure public strategy profile can be described as an automaton as follows: There is a set of states, $W$, an initial state, $w^1 \in W$, a transition function $\sigma : W \times \mathcal{Y} \rightarrow W$, and a collection of decision rules, $d_i : W \rightarrow A_i$. In the first period, player $i$ chooses action $a_1^i = d_i(w^1)$. The vector of actions, $a^1$, then generates a signal $y^1$ according to the distribution $\rho(\cdot|a^1)$. In the second period, player $i$ chooses the action $a_2^i = d_i(w^2)$, where $w^2 = \sigma(w^1,y^1)$, and so on. Since we can take $W$ to be the set of all histories of the public signal, $\cup_{k\geq 0} \mathcal{Y}^k$, $W$ is at most countably infinite. A public profile is finite if $W$ is a finite set. Note that, given a pure strategy profile (and the associated automaton), continuation play after any history is determined by the public state reached by that history. In games with private monitoring, by contrast, given an action-free strategy profile (and the associated automaton), a sufficient statistic for continuation play after any history is the vector of current private states, one for each player.

Denote the vector of average discounted expected values of following the public profile $(W,w,\sigma,d)$ (i.e., the initial state is $w$) by $\phi(w)$. Define a function $g : A \times W \rightarrow W$ by $g(a;w) \equiv (1-\delta)w(a) + \delta \sum_y \phi(\sigma(w,y)) \rho(y|a)$. We have (from Abreu, Pearce, and Stacchetti (1990)), that if the profile is an equilibrium, then, for all $w \in W$, the action profile $(d_1(w),\ldots,d_N(w)) \equiv d(w)$ is a pure strategy equilibrium of the static game with strategy spaces $A_i$ and payoffs $g_i(\cdot;w)$ for each $i$ and, moreover, $\phi(w) = g(d(w),w)$. Conversely, if $(W,w^1,\sigma,d)$ describes an equilibrium of the static game with payoffs $g(\cdot;w)$ for all $w \in W$, then the induced pure strategy profile in the infinitely repeated game with public monitoring is an equilibrium.\footnote{Note that strategies of public-monitoring games are public if and only if they are action-free when we view the public-monitoring game as a game with (trivial) private monitoring.}

A PPE $(W,w^1,\sigma,d)$ is strict if, for all
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$w \in W$, $d(w)$ is a strict Nash equilibrium of the static game $g(\cdot; w)$.

A maintained assumption throughout our analysis is that the public monitoring has full support.

**Assumption 1** $\rho(y|a) > 0$ for all $y \in Y$ and all $a \in A$.

**Definition 2** An automaton is minimal if for every pair of states $w, \hat{w} \in W$, there exists a sequence of signals $y^1, \ldots, y^L$ such that for some $i$, $d_i(\sigma(y^1, \ldots, y^L; w)) \neq d_i(\sigma(y^1, \ldots, y^L; \hat{w}))$, where $\sigma(y^1, \ldots, y^L; w) \equiv \sigma(y^L, \sigma(..., \sigma(y^1, w)))$.

The restriction to minimal automata is without loss of generality: every profile has a minimal representing automaton.

3. Almost-Public Monitoring

3.1. Minimally-private almost-public monitoring

In an earlier paper (Mailath and Morris (2002)), we investigate the robustness of public perfect equilibria to minimal perturbations in the direction of private monitoring. Games with public monitoring $(Y, \rho)$ are nested within games with private monitoring, since public monitoring simply means that all players always observe the same signal, i.e., $\Omega_i = \Omega_j = Y$, and $\pi(y, \ldots, y|a) = \rho(y|a)$ for all $a$. Mailath and Morris (2002) discussed the case of minimally-private monitoring, in the sense that there is a public monitoring distribution $(Y, \rho)$ with $\Omega_i = Y$ and $\pi$ close to $\rho$.

**Definition 3** A private-monitoring game $(u^*, (Y^n, \pi))$ is $\varepsilon$-close to a public-monitoring game $(\hat{u}^*, (Y, \rho))$, if $|\hat{u}^i(y_i, a_i) - u^i_i(y, a_i)| < \varepsilon$ and $|\pi((y, \ldots, y)|a) - \rho(y|a)| < \varepsilon$ for all $i \in N$, $y \in Y$ and all $a \in A$. We also say that such a private-monitoring game has minimally-private almost-public monitoring.

Note that for $\eta > 0$ there is $\varepsilon > 0$ such that if $(u^*, (Y^n, \pi))$ is $\varepsilon$-close to $(\hat{u}^*, (Y, \rho))$, then $\left| \sum_{y_1, \ldots, y_n} u^i(y_i, a_i)\pi(y_1, \ldots, y_n|a) - \sum_{y} \hat{u}^i(y, a_i)\rho(y|a) \right| < \eta$. In other words, the ex ante stage payoffs of any minimally-private almost-public-monitoring game are close to the ex ante stage payoffs of the benchmark public-monitoring game.

An important implication of the assumption that the public monitoring is full support is that when a player observes a private signal $y$, then (for $\varepsilon$ small) that player assigns high probability to all other players also observing the same signal, irrespective of the actions taken.

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8 Equivalently, a PPE is strict if each player strictly prefers his equilibrium strategy to every other public strategy. For a large class of public-monitoring games, strictness is without loss of generality, in that the folk theorem holds for strict PPE (Fudenberg, Levine, and Maskin (1994, Theorem 6.4 and remark)).
Lemma 2  Fix a full support public monitoring distribution \( \rho \) and \( \eta > 0 \). There exists \( \varepsilon > 0 \) such that if \( \pi \) is \( \varepsilon \)-close to \( \rho \), then for all \( a \in A \),

\[
\pi_i(y_1|a,y) > 1 - \eta.
\]

A public strategy profile \((W,w^1,\sigma,d)\) in the public-monitoring game induces a strategy profile \((s_1,\ldots,s_n)\) in minimally-private almost-public-monitoring games in the obvious way: \( s^1 = d_i(w^1) \) in minimally-private almost-public-monitoring games, and defining states recursively by \( s^i_{t+1} = \sigma(w^i_{t},y^i_{t}) \). This private strategy is, of course, action-free.

If \( W \) is finite, each player can be viewed as following a finite state automaton. Hopefully without confusion, when we can take the initial state as given, we abuse notation and write \( w^i_t = \sigma(w^i_1,h^i_t) = \sigma(h^i_t) \). We describe \( w^i_t \) as player \( i \)'s private state in period \( t \). It is important to note that while all players are in the same private state in the first period, since the signals are private, after the first period, different players may be in different private states. The private profile is the translation to the private-monitoring game of the public profile (of the public-monitoring game).

If player \( i \) believes that the other players are following a strategy that was induced by a public profile, then a sufficient statistic for \( h^i_t \) is player \( i \)'s private state and \( i \)'s beliefs over the other players’ private states, i.e., \((w^i_t,\beta^i_t)\), where \( \beta^i_t \in \Delta(W^{N-1}) \). In principle, \( W \) may be quite large. For example, if the public strategy profile is nonstationary, it may be necessary to take \( W \) to be the set of all histories of the public signal, \( \cup_{k \geq 0} Y^k \). On the other hand, the strategy profiles typically studied can be described with a significantly more parsimonious collection of states, often finite. When \( W \) is finite, the need to only keep track of each player’s private state and that player’s beliefs over the other players’ private states is a considerable simplification, as the following result (Mailath and Morris (2002, Theorem 4.2)) demonstrates.

**Theorem 1** Suppose the public profile \((W,w^1,\sigma,d)\) is a strict equilibrium of the full-support public-monitoring game for some \( \delta \) and \( |W| < \infty \). For all \( \kappa > 0 \), there exists \( \eta \) and \( \varepsilon \) such that in any game with minimally-private almost-public monitoring, if the posterior beliefs induced by the private profile satisfy \( \beta^i_t(\sigma(h^i_t) \mid 1) > 1 - \eta \) for all \( h^i_t = (d_i(w^1),y^i_1;\ldots;d_i(w^i_{t-1}),y^i_{t-1}) \), where \( w^i_{t+1} \equiv \sigma(w^i_t,y^i_t) \), and if \( \pi \) is \( \varepsilon \)-close to \( \rho \), then the private profile is a Nash equilibrium of the game with private monitoring for the same \( \delta \) and the expected payoff in that equilibrium is within \( \kappa \) of the public equilibrium payoff.

**Example 2** We return to the repeated prisoners’ dilemma, with ex ante stage game given by (1) (recall footnote 5). In the benchmark public-monitoring game, the set of
public signals is $Y = \{y, \bar{y}\}$ and public monitoring distribution is

$$\rho(y|a_1a_2) = \begin{cases} p, & \text{if } a_1a_2 = e_1e_2 \\ q, & \text{if } a_1a_2 = e_1n_2 \text{ or } n_1e_2, \\ r, & \text{if } a_1a_2 = n_1n_2. \end{cases}$$

The grim trigger strategy profile for the public-monitoring game is described by the automaton $W = \{w^e, w^n\}$, initial state $w^e$, decision rules $d_i(w^a) = a_i$, and transition rule

$$\sigma(w, y) = \begin{cases} w^e, & \text{if } y = \bar{y} \text{ and } w = w^e, \\ w^n, & \text{otherwise.} \end{cases}$$

Grim trigger is a strict PPE if $\delta > (3p - 2q)^{-1} > 0$ (a condition we maintain throughout this example). We turn now to minimally-private-monitoring games that are $\varepsilon$-close to this public-monitoring game. It turns out that, for $\varepsilon$ small, grim trigger induces a Nash equilibrium in such games if $q < r$, but not if $q > r$. Consider first the case $q > r$ and the private history $(e_1y_1, n_1\bar{y}_1, n_1y_1, \ldots, n_1\bar{y}_1)$. We now argue that, after a sufficiently long such history, the grim trigger specification of $n_1$ is not optimal. Intuitively, while player 1 has transited to the private state $w^e_1$, player 1 always puts strictly positive (but perhaps small) probability on his opponent being in private state $w^e_2$. Since $q > r$ (and $\varepsilon$ is small), the private signal $\bar{y}_1$ after playing $n_1$ is an indication that player 2 had played $e_2$ (rather than $n_2$), and so player 1’s posterior that player 2 is still in $w^e_2$ increases. Eventually, player 1 is sufficiently confident of player 2 still being in $w^e_2$ that he finds $n_1$ suboptimal. On the other hand, when $q \leq r$, such a history is not problematic because it reinforces 1’s belief that 2 is also in $w^n_2$. Two other histories are worthy of mention: $(e_1y_1, n_1y_1, n_1\bar{y}_1, \ldots, n_1y_1)$ and $(e_1\bar{y}_1, e_1\bar{y}_1, e_1\bar{y}_1, \ldots, e_1\bar{y}_1)$. Under the first history, while the signal $y_2$ is now a signal that 2 had chosen $e_2$ in the previous period, for $\varepsilon$ small, 1 is confident that 2 also observed $y_2$ and so will transit to $w^n_2$. For the final history, the signal $\bar{y}_1$ continually reassures 1 that 2 is still playing $e_2$, and so $e_1$ remains optimal. (See Mailath and Morris (2002, Section 3.3) for the calculations underlying this discussion.)

**Example 3** As the players become patient, the payoffs from grim trigger converge to $(0,0)$. A grim trigger profile (i.e., a profile in which the specification of $n_i$ is absorbing) can only achieve significant payoffs for patient players by being forgiving.9 Such a profile provides a different example of how a strict PPE can fail to induce a Nash equilibrium in close-by minimally-private-monitoring games. The simplest forgiving profile requires two realizations of $y$ to switch to $n_1n_2$. The automaton for this profile has a set of

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9 This is the class of profiles studied by Compte (2002) for the conditionally-independent private-monitoring prisoners’ dilemma.
Figure 2: Forgiving grim trigger where any two realizations of $y$ lead to $w^n$.

states $W = \{w^e, \hat{w}^e, w^n\}$, initial state $w^e$, decision rules $d_i(w^a) = a_i$ and $d_i(\hat{w}^e) = e_i$, and transition function

$$\sigma(w, y) = \begin{cases} 
w^e, & \text{if } y = \bar{y} \text{ and } w = w^e, \\
\hat{w}^e, & \text{if } y = y \text{ and } w = w^e \text{ or } y = \bar{y} \text{ and } w = \hat{w}^e, \\
w^n, & \text{otherwise.} 
\end{cases}$$

The profile is illustrated in Figure 2. This PPE never induces a Nash equilibrium in close-by minimally-private-monitoring games: consider a private history in which player 1 plays $e_1$ and observes $\bar{y}_1$ for $T$ periods, and then observes $y_1$. Under the forgiving profile, player 1 is supposed switch to the private state $\hat{w}^e_1$ and continue to play $e_1$ (until another $y_1$ is observed). But, for large $T$, it is more likely that player 2 has observed $y_2$ in exactly one of the first $T$ periods than having observed $\bar{y}_2$ in every period. Consequently, for large $T$, player 1 will not find $e_1$ optimal. Clearly, the same analysis applies to forgiving grim triggers that require more realizations of $y$ to switch to $w^n$.

Another class of forgiving grim trigger profiles require successive realizations of $y$ to switch to $w^n$. In the three state version, the automaton is identical to that above except $\sigma(\hat{w}^e, \bar{y}) = w^e$ (see Figure 3). The analysis of this profile is similar to that of Example 2. The profile does not induce a Nash equilibrium in close-by minimally-private-monitoring games if $q > r$ for similar reasons. There are now two possibilities for the case $q \leq r$, since isolated observations of $y_1$ do not lead to $w^o_2$. For the histories considered in Example 2, the same argument applies once we note that, conditional on players being in one of $w^e$ or $\hat{w}^e$, a player assigns very high probability to the other player being in the same state, since this is determined by the last signal. The remaining histories are those with isolated observations of $y_1$. The critical history (since it contains the largest fraction of $y_1$’s consistent with $e_1$) is $(e_1\bar{y}_1, e_1y_1, e_1\bar{y}_1, e_1y_1, \ldots, e_1\bar{y}_1)$, that is, alternating

\footnote{This type of drift of beliefs is a general phenomenon when players choose the same action in adjacent states (see also Example 6).}
Figure 3: Forgiving grim trigger where two successive realizations of $y$ lead to $w^n$.

$y_1$ and $\tilde{y}_1$. If $p(1 - p) \geq q(1 - q)$, then such a history (weakly) indicates that player 2 is still playing $e_2$, while the reverse strict inequality indicates that player 2 is playing $n_2$. Summarizing, the profile induces a Nash equilibrium in close-by minimally-private-monitoring games if and only if $q \leq r$ and $p(1 - p) \geq q(1 - q)$.

3.2. General almost-public monitoring

We now turn to more general private monitoring structures that nonetheless preserve the essential characteristics of both Definition 3 and Lemma 2.11.

**Definition 4** The private monitoring distribution $(\Omega, \pi)$ is $\varepsilon$-close to the public monitoring distribution $(Y, \rho)$ if there exist signaling functions $f_i : \Omega_i \rightarrow Y \cup \{\emptyset\}$ such that

1. for each $a \in A$ and $y \in Y$,

$$\left| \pi(\{\omega : f_i(\omega_i) = y \text{ for all } i\} | a) - \rho(y | a) \right| \leq \varepsilon,$$

and

2. for all $y \in Y$, $\omega_i \in f_i^{-1}(y)$, and all $a \in A$,

$$\pi(\{\omega_{-i} : f_j(\omega_j) = y \text{ for all } j \neq i\} | (a, \omega_i)) \geq 1 - \varepsilon.$$

11While there is a connection to informational smallness (see, for example, McLean and Postlewaite (forthcoming)), these are distinct notions. For concreteness, suppose $\omega_i$ is a noisy signal of $y$. Then, $(\Omega, \pi)$ is $\varepsilon$-close to $(Y, \rho)$ if and only if the private signal is a sufficiently accurate signal of $y$. A player is informational small if the posterior on $y$, conditional on the other players’ private signals, on average does not vary too much with that player’s private signal. Even if each player’s private signal is very accurate, the posterior can vary drastically in a player’s signal if that player’s signal is sufficiently accurate relative to the other players. Moreover, if there are many players, even when signals are very noisy, each player will be informationally small.
The private monitoring distribution \((\Omega, \pi)\) is strongly \(\varepsilon\)-close to the public monitoring distribution \((Y, \rho)\) if it is \(\varepsilon\)-close, and in addition, all the signaling functions map into \(Y\).

If the private monitoring is \(\varepsilon\)-close, but not strongly \(\varepsilon\)-close, then some private signals are not associated with any public signal: there is a signal \(\omega_i\) satisfying \(f_i(\omega_i) = \emptyset\). Such an "uninterpretable" signal may contain no information about the signals observed by the other players.

Note that the second condition implies that every player has at least one private signal mapped to each public signal. Moreover, for the case \(\Omega_i = Y\), the first condition implies the second (Lemma 2).

The condition of \(\varepsilon\)-closeness in Definition 4 can be restated as follows. Recall from Monderer and Samet (1989) that an event is \(p\)-evident if, whenever it is true, everyone assigns probability at least \(p\) to it being true.

**Lemma 3** The private monitoring distribution \((\Omega, \pi)\) is \(\varepsilon\)-close to the public monitoring distribution \((Y, \rho)\) if and only if there are signaling functions \(f_i : \Omega_i \to Y \cup \{\emptyset\}\) such that for each public signal \(y\), the set of private signal profiles \(\{\omega : f_i(\omega_i) = y \text{ for all } i\}\) is \((1 - \varepsilon)\)-evident (conditional on any action profile) and has probability within \(\varepsilon\) of the probability of \(y\) (conditional on that action profile).

**Example 4** We now allow player 1 to have a richer set of private signals, \(\Omega_1 = \{y_1, y'_1, y''_1\}\), keeping player 2 signals unchanged, \(\Omega_2 = \{y_2, y_2\}\). For the action profile \(a_1a_2\), the joint probabilities are given by:

<table>
<thead>
<tr>
<th>(a_1a_2)</th>
<th>(y_2)</th>
<th>(y_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y_1)</td>
<td>((1 - \alpha)(1 - 3\varepsilon))</td>
<td>(\varepsilon)</td>
</tr>
<tr>
<td>(y'_1)</td>
<td>(\varepsilon)</td>
<td>(\alpha'(1 - 3\varepsilon))</td>
</tr>
<tr>
<td>(y''_1)</td>
<td>(\varepsilon)</td>
<td>((\alpha - \alpha')(1 - 3\varepsilon))</td>
</tr>
</tbody>
</table>

where \(\alpha = p\) if \(a_1a_2 = e_1e_2\), \(q\) if \(a_1a_2 = e_1n_2\) or \(n_1e_2\), and \(r\) if \(a_1a_2 = n_1n_2\). This private-monitoring distribution is \(\varepsilon\)-close to the public-monitoring distribution of Example 2 using the signaling functions \(f_1(y_1) = y\) and \(f_2(y_2) = f_1(y'_1) = f_1(y''_1) = y\). Note that even for \(\varepsilon\) small, the only restriction on the values of \(p', q', \text{ and } r'\) is that they be smaller than \(p, q, \text{ and } r\) (respectively).

**Definition 5** A private-monitoring game \((u^*, (\Omega, \pi))\) is \(\varepsilon\)-close to a public-monitoring game \((\tilde{u}^*, (Y, \rho))\), if \((\Omega, \pi)\) is \(\varepsilon\)-close to \((Y, \rho)\) (with associated signaling functions \((f_1, \ldots, f_n)\)) and \(|\tilde{u}_i^*(f_i(\omega_i), a_i) - u_i^*(\omega_i, a_i)| < \varepsilon\) for all \(i \in N\), \(a_i \in A_i\), and \(\omega_i \in f_i^{-1}(y)\). We will also say that such a private-monitoring game has almost-public monitoring.
As above, the ex ante stage payoffs of any almost-public-monitoring game are close to the ex ante stage payoffs of the benchmark public-monitoring game.

**Lemma 4** For all \( \eta > 0 \), there is \( \varepsilon > 0 \) such that if \((u^*, (\Omega, \pi)) \) is \( \varepsilon \)-close to \((\hat{u}^*, (Y, \rho))\), then \( \left| \sum_{\omega_1, \ldots, \omega_n} u^*_i(\omega_i, a_i) \pi(\omega_1, \ldots, \omega_n|a) - \sum_{y} \hat{u}^*_i(y, a_i) \rho(y|a) \right| < \eta \).

Fix a public profile \((W, w^1, \sigma, d)\) of a full-support public-monitoring game \((\hat{u}^*, (Y, \rho))\), and a strongly \( \varepsilon \)-close private-monitoring game \((u^*, (\Omega, \pi))\). The public profile induces a private profile in the private-monitoring game in a natural way: Player \( i \)'s strategy is described by the automaton \((W, w^1, \sigma_i, d_i)\), where \( \sigma_i(w, \omega_i) = \sigma(w, f_i(\omega_i)) \) for all \( \omega_i \in \Omega_i \) and \( w \in W \). The set of states, initial state, and decision function are from the public profile. The transition function \( \sigma_i \) is well-defined, because the signaling functions all map into \( Y \), rather than \( Y \cup \{\emptyset\} \). As for games with minimally-private almost-public monitoring, if player \( i \) believes that the other players are following a strategy induced by a public profile, a sufficient statistic for any private history \( h_i^t \) is player \( i \)'s private state and \( i \)'s beliefs over the other players’ private states, i.e., \((w_i^1, \beta_i^1)\), where \( \beta_i^1 \in \Delta(W^{N-1}) \). Finally, we can recursively calculate the private states of player \( i \) as \( w_i^2 = \sigma(w^1, f_i(\omega_i^1)) = \sigma_i(w^1, \omega_i^1) \), \( w_i^3 = \sigma_i(w_i^2, \omega_i^2) \), and so on. Thus, for any private history \( h_i^t \), we can write \( w_i^t = \sigma_i(h_i^t) \).

**Example 5** In Example 2, we argued that if \( q < r \), grim trigger induces Nash equilibrium behavior in close-by minimally-private-monitoring games. We now argue that under the private monitoring distribution of Example 4, even if \( q < r \), grim trigger will not induce a Nash equilibrium behavior in some close-by games. In particular, suppose \( 0 < r' < q' < q < r \). Under this parameter restriction, the signal \( y_i^n \) after \( n_1 \) is indeed a signal that player 2 had also played \( n_2 \). However, the signal \( y_i^n \) after \( n_1 \) is a signal that player 2 had played \( e_2 \) and so a sufficiently long private history of the form \((e_1 y_1, n_1 y_i^1, n_1 y_i^2, \ldots, n_1 y_i^n)\) will lead to a posterior for player 1 at which \( n_1 \) is not optimal.

### 4. PPE with bounded recall

As we saw in Example 5, arbitrary public equilibria need not induce equilibria of almost-public-monitoring games, because the public state in period \( t \) is determined, in principle, by the entire history \( h_i^t \). For profiles that have bounded recall, the entire history is not needed, and equilibria in bounded recall strategies will induce equilibria in almost-public-monitoring games.\(^{12}\)

\(^{12}\) Mailath and Morris (2002) used the term bounded memory for public profiles with the property that there is an integer \( L \) such that a representing automaton is given by \( W = (Y \cup \{\ast\})_L \), \( \sigma(y, (y^L, \ldots, y^2, y^1)) = (y, y^L, \ldots, y^2) \) for all \( y \in Y \), and \( w^1 = (\ast, \ldots, \ast) \). Our earlier notion im-
Definition 6 A public profile \( s \) has bounded recall if there exists \( L \) such that for all \( h^t = (y_1, \ldots, y^{t-1}) \) and \( \hat{h}^t = (\hat{y}_1, \ldots, \hat{y}^{t-1}) \), if \( t > L \) and \( y^\tau = \hat{y}^\tau \) for all \( \tau = t - L, \ldots, t - 1 \), then
\[ s(h^t) = s(\hat{h}^t). \]

The following characterization of bounded recall is useful.

Lemma 5 The public profile induced by the minimal automaton \((W, w^1, \sigma, d)\) has bounded recall if and only if there exists \( L \) such that for all \( w, w' \in W \) reachable in the same period and for all \( h \in Y^\infty \),
\[ \sigma(w, h^L) = \sigma(w', h^L). \]

Fix a strict public equilibrium with bounded recall, \((W, w^1, \sigma, d)\). Fix a private monitoring technology \((\Omega, \pi)\) with associated signaling functions \( f_i \) that is \( \varepsilon \)-close to \((Y, \rho)\). Following Monderer and Samet (1989), we first consider a constrained game where behavior after “uninterpretable signals” is arbitrarily fixed. Define the set of “uninterpretable” private histories, \( H^u_i \), \(\{h_i^t : \omega_i^t \in f_i^{-1}(\emptyset), \text{some } \tau \text{ satisfying } t - L \leq \tau \leq t - 1\}\). This is the set of private histories for which in any of the last \( L \) periods, a private signal \( \omega_i^\tau \) satisfying \( f_i(\omega_i^\tau) = \emptyset \) is observed. We fix arbitrarily player \( i \)'s action after any private history \( h_i^t \in H^u_i \). For any private history that is not uninterpretable, each of the last \( L \) observations of the private signal can be associated with a public signal by the function \( f_i \). Denote by \( w_i(h_i^t) \) the private state so obtained. That is,
\[ w_i(h_i^t) = (f_i(\omega_i^{t-1}), \ldots, f_i(\omega_i^{t-L})), \]
for all \( h_i^t \not\in H^u_i \). We are then left with a game in which in period \( t \geq 2 \) player \( i \) only chooses an action after a signal \( \omega_i^{t-1} \) yields a private history not in \( H^u_i \). We claim that for \( \varepsilon \) sufficiently small, the profile \((\hat{s}_1, \ldots, \hat{s}_N)\) is an equilibrium of this constrained game, where \( \hat{s}_i \) is the strategy for player \( i \):
\[ \hat{s}_i(h_i^t) = \begin{cases} d_i(w_i^1), & \text{if } t = 1, \\ d_i(w_i(h_i^t)), & \text{if } t > 1 \text{ and } h_i^t \not\in H^u_i. \end{cases} \]

But this follows from arguments almost identical to that in the proofs of Mailath and Morris (2002, Theorems 4.2 and 4.3): since a player’s behavior depends only on the last \( L \) signals, for small \( \varepsilon \), after observing a history \( h_i^t \not\in H^u_i \), player \( i \) assigns a high probability to player \( j \) observing a signal that leads to the same private state. The crucial point is that for \( \varepsilon \) small, the specification of behavior after signals \( \omega_i \) satisfying \( f_i(\omega_i) = \emptyset \) is irrelevant for behavior at signals \( \omega_i \) satisfying \( f_i(\omega_i) \in Y \). It remains implicitly imposes a time homogeneity condition, since the caveat in Lemma 5 that the two states should be reachable in the same period is missing. The strategy profile in which play alternates between the same two action profiles in odd and even periods has bounded recall, but not bounded memory.
to specify optimal behavior after signals $\omega_i$ satisfying $f_i(\omega_i) = \emptyset$. So, consider a new constrained game where player $i$ is required to follow $\hat{s}_i$ where possible. This constrained game has an equilibrium, and so by construction, we thus have an equilibrium of the unconstrained game. We have thus proved:

**Theorem 2** Fix a full-support public-monitoring game $(\tilde{u}^*, (Y, \rho))$ and a strict bounded recall public perfect equilibrium. There exists $\varepsilon > 0$ such that for all private-monitoring games $(u^*, (\Omega, \pi)) \varepsilon$-close to $(\tilde{u}^*, (Y, \rho))$,

1. if $f_i(\Omega_i) = Y$ for all $i$, the induced private profile is a Nash equilibrium; and
2. if $f_i(\Omega_i) \neq Y$ for some $i$, there is an extension of the profile to “uninterpretable” histories that is a Nash equilibrium.

We could similarly extend our results on patiently strict connected finite public profiles (Mailath and Morris (2002, Theorem 5.1)) and on the almost-public almost-perfect folk theorem (Mailath and Morris (2002, Theorem 6.1)) to this more general notion of nearby private-monitoring distributions.

## 5. Failure of Coordination

Example 5 illustrates that the updating in almost-public monitoring games can be very different than would be expected from the underlying public-monitoring game. In this section, we build on that example to show that when the set of signals is sufficiently rich (in a sense to be defined), many profiles fail to induce equilibrium behavior in almost-public monitoring games.

Our negative results are based on the following converse to Theorem 1 (the proof is in the Appendix). Since the theorem is negative, the assumption of strong $\varepsilon$-closeness makes the result more useful, not less.\(^{13}\)

**Theorem 3** Suppose the public profile $(W, w^1, \sigma, d)$ is a strict equilibrium of the full-support public-monitoring game $(\tilde{u}^*, (Y, \rho))$ for some $\delta$ and $|W| < \infty$. There exists $\eta > 0$ and $\varepsilon > 0$ such that for any game with private monitoring $(u^*, (\Omega, \pi))$ strongly $\varepsilon$-close to $(\tilde{u}^*, (Y, \rho))$, if there exists a player $i$, a private history for that player $h^1_i$, and

\(^{13}\)While we have stated this theorem, and Theorem 4 below, for pure strategies, they also hold for some mixed strategy profiles. Recall from Section 2 that given an automaton $(W, d_i, \sigma_i)$ describing a collection of pure strategies for player $i$ (taking any state $w \in W$ as the initial state gives a pure strategy), a probability distribution over $W$ gives a mixed strategy. Consider now a mixed strategy PPE of the game with public-monitoring. Clearly, such a profile cannot be strict. However, there may exist a period $T$, such that all the incentive constraints after period $T$ are strict. In that case, Theorem 3 holds if the hypotheses are satisfied for $t \geq T$. 
a state \( w \) such that \( d_i(w) \neq d_i(\sigma_i(h_i^t)) \) and \( \beta_i(w1|h_i^t) > 1 - \eta \), then the induced private profile is not a Nash equilibrium of the game with private monitoring for the same \( \delta \).

We implicitly used this result in our discussions of the repeated prisoners’ dilemma. For example, in Example 5, we argued that there was a private history for player 1 that leaves him in the private state \( w_1^0 \), but his posterior after that history assigns probability close to 1 that player 2’s private state is \( w_2^0 \).

Our approach is to ask when is it possible to so “manipulate” a player’s beliefs through selection of private history that the hypotheses of Theorem 3 are satisfied. In particular, we are interested in the weakest independent conditions on the private-monitoring distributions and on the strategy profiles that would allow such manipulation.

Fix a PPE of the public-monitoring game and a close-by almost-public-monitoring game. The logic of Example 5 runs as follows: Consider a player \( i \) in a private state \( \hat{w} \) who assigns strictly positive (albeit small) probability to all the other players being in some other common private state \( \bar{w} \neq \hat{w} \) (full-support private monitoring ensures that such an occurrence arises with positive probability). Let \( \bar{a} = (d_i(\hat{w}), d_{-i}(\bar{w})) \) be the action profile that results when \( i \) is in state \( \hat{w} \) and all the other players are in state \( \bar{w} \). Suppose that if any other player is in a different private state \( w \neq \bar{w} \), then the resulting action profile differs from \( \bar{a} \). Suppose, moreover, there is a signal \( y \) such that \( \hat{w} = \sigma(\hat{w}, y) \) and \( \bar{w} = \sigma(\bar{w}, y) \), that is, any player in the state \( \hat{w} \) or \( \bar{w} \) observing a private signal consistent with \( y \) stays in that private state (and so the profile cannot have bounded recall, see Lemma 5). Suppose finally there is a private signal \( \omega_i \) for player \( i \) consistent with \( y \) that is more likely to have come from \( \hat{a} \) than any other action profile, i.e., \( \omega_i \in f^{-1}_i(y) \) and (where \( \pi(\omega_i|a) \) is the probability that player \( i \) observes the signal \( \omega_i \) under \( a \))

\[
\pi(\omega_i|\hat{a}) > \pi(\omega_i|(d_i(\hat{w}), a'_{-i})) \quad \forall a'_{-i} \neq d_{-i}(\hat{w}).
\]  

Then, after observing the private signal \( \omega_i \), player \( i \)’s posterior probability that all the other players are in \( \bar{w} \) should increase (this is not immediate, however, since the monitoring is private). Moreover, since players in \( \hat{w} \) and \( \bar{w} \) do not change their private states, we can make player \( i \)’s posterior probability that all the other players are in \( \bar{w} \) as close to one as we like. If \( d_i(\hat{w}) \neq d_i(\bar{w}) \), an application of Theorem 3 shows that the induced private profile is not an equilibrium.

The suppositions in the above logic can be weakened in two ways. First, it is not necessary that the same private signal \( \omega_i \) be more likely to have come from \( \hat{a} \) than any other action profile. It should be enough if for each action profile different from \( \hat{a} \), there is a private signal more likely to have come from \( \hat{a} \) than that profile, as long as the signal not mess up the other inferences too badly. In that case, realizations of the other signals could undo any damage done without negatively impacting on the overall
inferences. For example, suppose there are two players, with player 1 the player whose beliefs we are “manipulating,” and in addition to state \( \bar{w} \), player 2 could be in state \( \bar{w} \) or \( w \). We would like the odds ratio \( \Pr(\bar{w}_2 \neq \bar{w} | h_1^i) / \Pr(\bar{w}_2 = \bar{w} | h_1^i) \) to converge to zero as \( t \to \infty \), for appropriate private histories. Let \( \bar{a}_1 = d_1(\bar{w}) \), \( \bar{a}_2 = d_2(\bar{w}) \), \( a'_2 = d_2(w) \), and \( a''_2 = d_2(w) \), and suppose there are two private signals, \( \omega'_1 \) and \( \omega''_1 \), satisfying

\[
\pi(\omega'_1 | \bar{a}_1, a''_2) > \pi(\omega'_1 | \bar{a}) > \pi(\omega'_1 | \bar{a}_1, a'_2)
\]

and

\[
\pi(\omega''_1 | \bar{a}_1, a''_2) > \pi(\omega''_1 | \bar{a}) > \pi(\omega''_1 | \bar{a}_1, a'_2).
\]

Then, after observing the private signal \( \omega'_1 \), we have

\[
\frac{\Pr(\bar{w}_2 = \bar{w} | h_1^i, \omega'_1)}{\Pr(\bar{w}_2 = \bar{w} | h_1^i, \omega'_1)} = \frac{\pi(\omega'_1 | \bar{a}, a'_2) \Pr(\bar{w}_2 = \bar{w} | h_1^i)}{\pi(\omega'_1 | \bar{a}) \Pr(\bar{w}_2 = \bar{w} | h_1^i)} < \frac{\Pr(\bar{w}_2 = \bar{w} | h_1^i)}{\Pr(\bar{w}_2 = \bar{w} | h_1^i)}
\]

as desired, but \( \Pr(w_2 = w | h_1^i, \omega'_1) / \Pr(\bar{w}_2 = \bar{w} | h_1^i, \omega'_1) \) increased. On the other hand, after observing the private signal \( \omega''_1 \), while the odds ratio \( \Pr(w_2 = w | h_1^i, \omega''_1) / \Pr(\bar{w}_2 = \bar{w} | h_1^i, \omega''_1) \) falls, \( \Pr(w_2 = \bar{w} | h_1^i, \omega''_1) / \Pr(w_2 = \bar{w} | h_1^i, \omega'_1) \) increases. However, it may be that the increases can be offset by appropriate decreases, so that, for example, \( \omega'_1 \) followed by two realizations of \( \omega''_1 \) results in a decrease in both odds ratios. If so, a sufficiently high number of realizations of \( \omega'_1 \omega''_1 \omega''_1 \) result in \( \Pr(w_2 \neq \bar{w} | h_1^i) / \Pr(w_2 = \bar{w} | h_1^i) \) being close to zero.

Our richness condition on private monitoring distributions captures this idea. We consider sequences of private monitoring distributions \((\Omega, \pi^k)\), where \((\Omega, \pi^k)\) is strongly 1/k-close to \((Y, \rho)\) (note that \( \Omega \) is independent of \( k \)). Define \( \gamma^k_{a_a} (\omega_i) \equiv \log \pi^k(\omega_i | a_i, a_{-i}^-) - \log \pi^k(\omega_i | a_i, a_{-i}^+) \), and let \( \gamma^k_a (\omega_i) = \left( \gamma^k_{a_a} (\omega_i) \right)_{a_a \in A_a, a_{-a} \neq a_{-a}} \) denote the vector in \( \mathbb{R}^{n_A-1} \) of the log odds ratios of the signal \( \omega_i \) associated with different action profiles.

**Definition 7** A sequence of private monitoring distributions \(\{\Omega, \pi^k\}_k\) is rich if for all \((a, \omega) \in A \times \Omega, \{\pi^k(\omega | a)\}_k\) is a convergent sequence with limit \( \pi^\infty(\omega | a) \), if for all \((a, \omega_i) \in A \times \Omega_i, \pi^\infty(\omega_i | a) > 0 \), and if for all \( y \in Y \), the convex hull of the set of vectors \(\{\gamma^\infty_a (\omega_i) : \omega_i \in f_y^{i-1} (y)\} \) has a nonempty intersection with \( \mathbb{R}^{n_A-1} \).

The second weakening concerns the nature of the strategy profile. The logic assumed that there is a signal \( y \) such that \( \bar{w} = \sigma(\bar{w}, y) \) and \( \bar{w} = \sigma(\bar{w}, y) \). If there were only two states, \( \bar{w} \) and \( \bar{w} \), it would clearly be enough that there be a finite sequence of signals such that both \( \bar{w} \) and \( \bar{w} \) cycle. When there are more states, we also need to worry about what happens to the other states. In addition, we need to allow for time-dependent profiles, and profiles that use some states for only a finite time. Let \( W_t \) be the set of states reachable in period \( t, W_t = \{w \in W : w = \sigma(w^1, y^1, y^2, \ldots, y^{t-1}) \} \) for some
\((y_1, y_2, \ldots, y_{t-1})\), where \(w^1\) is the initial state\}. Define \(R(\tilde{w})\) as the set of states that are repeatedly reachable in the same period as \(\tilde{w}\) (i.e., \(R(\tilde{w}) = \{w \in W : \{w, \tilde{w}\} \subset W_t\ \text{infinitely often}\}\)).

We generalize the cycling idea to the notion that there be a path that allows some distinguished state to be *separated* from every other state that could ever be reached. Given an outcome path \(h \equiv (y_1, y_2, \ldots) \in Y^\infty\), let \(\tau h \equiv (y_\tau, y_{\tau+1}, \ldots) \in Y^\infty\) denote the outcome path from period \(\tau\), so that \(h = (h^\tau, \tau h)\) and \(\tau h^{\tau+t} = (y_\tau, y_{\tau+1}, \ldots, y_{\tau+t-1})\).

**Definition 8** The public strategy profile is separating if there is some state \(\tilde{w}\) and an outcome path \(h \in Y^\infty\) such that there is another state \(w \in R(\tilde{w})\) that satisfies \(\sigma(w, h^t) \neq \sigma(\tilde{w}, h^t)\) for all \(t\), and for all \(\tau\) and \(w \in R(\sigma(\tilde{w}, h^\tau))\), if \(\sigma(w, \tau h^{\tau+t}) \neq \sigma(\tilde{w}, \tau h^{\tau+t})\) for all \(t \geq 0\), then

\[
\sigma_i(w, \tau h^{\tau+t}) \neq \sigma_i(\tilde{w}, \tau h^{\tau+t}) \quad \text{infinitely often, for all } i.
\]

When the set of states is finite, separation turns out to imply a seemingly stronger cycling condition (see Lemma 6). Clearly, a separating profile cannot have bounded recall. Moreover, it is easy to construct PPE that neither have bounded recall nor are separating (Example 6). Nonetheless, we are not aware of any strict PPE of substantive interest that neither have bounded recall nor are separating.

**Example 6** The stage game is

\[
\begin{array}{ccc}
A & B & C \\
A & 3, 3 & 0, 0 & 0, 0 \\
B & 0, 0 & 3, 3 & 0, 0 \\
C & 0, 0 & 0, 0 & 2, 2 \\
\end{array}
\]

and in the public-monitoring game, there are two public signals, \(y'\) and \(y''\), with distribution \((0 < q < p < 1)\)

\[
\rho(y''|a_1a_2) = \begin{cases} 
p, & \text{if } a_1 = a_2, \\
q, & \text{otherwise.} \end{cases}
\]

The profile illustrated in Figure 4 is not separating, and yet is not robust. After enough realizations of private signals corresponding to \(y''\), beliefs must assign roughly equal probability to \(w^A\) and \(\tilde{w}^A\),\(^{14}\) and so after the first realization of a private signal corresponding to \(y'\), \(B\) is the only best reply (even if the current state is \(w^C\)). This example

\(^{14}\text{This is most easily seen by considering the Markov chain describing player 2’s private state transitions conditional on player 1 always playing } A \text{ and always observing the same private signal consistent with } y'' (a Markov chain is associated with each } \omega \in f_1(y'')\). Each such Markov chain is ergodic, and so has a unique stationary distribution. A straightforward calculation shows that, in the limit (as the private-monitoring distributions become arbitrarily close), the probability assigned to \(w^A_2\) equals \(\frac{1}{2}\).
Figure 4: In states $w^A$ and $\hat{w}^A$, the action $A$ is played, while in $w^B$ the action $B$ and in $w^C$, the action $C$ is played. This profile is not separating: the only cycle in which two states appear is $y''y''$.

(like the second forgiving grim trigger of Example 3) illustrates the possibility that beliefs over private states can drift to a stationary distribution when play is identical in different states.

**Theorem 4** Fix a separating strict finite PPE of a public-monitoring game $(\tilde{u}^*, (Y, \rho))$. Suppose $\{(u^k, (\Omega, \pi^k))\}$ is a sequence of private-monitoring games, with $(u^k, (\Omega, \pi^k))$ strongly $1/k$-close to $(\tilde{u}^*, (Y, \rho))$ and $\{(\Omega, \pi^k)\}$ a rich sequence of distributions. For $k$ sufficiently large, the induced private profile is not a Nash equilibrium of the private monitoring game.

Thus, separating strict PPE of public-monitoring games are not robust to the introduction of private monitoring.\(^{15}\) It, of course, also implies that separating behavior in the private-monitoring game typically cannot coordinate continuation play in the following sense. Say a profile is $\varepsilon$-strict if all the incentive constraints are satisfied by at least $\varepsilon$. (The result follows immediately from upperhemicontinuity and Theorem 4.)

**Corollary 1** Suppose $\{(u^k, (\Omega, \pi^k))\}$ is a sequence of private-monitoring games, with $(u^k, (\Omega, \pi^k))$ $1/k$-close to some public-monitoring game $(\tilde{u}^*, (Y, \rho))$ and $\{(\Omega, \pi^k)\}$ a rich sequence of distributions. Fix a pure strategy profile of the private monitoring game

---

\(^{15}\)The extension to mixed strategies described in footnote 13 also holds for Theorem 4.
in which each player’s strategy respects his signaling function \( f_i \) (i.e., \( \sigma_i(h_i, a_i, \omega_i) = \sigma_i(h_i, a_i, \hat{\omega}_i) \) if \( f_i(\omega_i) = f_i(\hat{\omega}_i) \neq \emptyset \)). Suppose this profile is separating (when interpreted as a public profile). For all \( \varepsilon > 0 \), there exists \( k' \) such that for \( k > k' \), this profile is not an \( \varepsilon \)-strict Nash equilibrium.

Since the equilibrium failure of separating profiles seem to arise after private histories that have low probability, an attractive conjecture is that equilibrium can be restored by appropriately modifying the profile at only the problematic histories. Unfortunately, such a modification, would appear to require additional modifications to the profile, destroying the connection to the public-monitoring game.

6. The Proof of Theorem 4

We present the proof through a series of Lemmas. We first show that separating profiles can be treated as if they cycle under the separating outcome path.

**Lemma 6** For any finite separating public strategy profile of the public-monitoring game, there is a finite sequence of signals \( \bar{y}^1, \ldots, \bar{y}^m \) and a state \( \bar{w} \) such that

1. every state in \( W_c \equiv \{ \sigma(w, \bar{y}^1, \ldots, \bar{y}^M) : w \in R(\bar{w}) \} \) cycles under the sequence of signals \( \bar{y}^1, \ldots, \bar{y}^m \) (i.e., \( \sigma(w, \bar{y}^1, \ldots, \bar{y}^m) = w \) for all \( w \in W_c \)),

2. \( \forall w \in W_c \setminus \{ \bar{w} \} \) \( \forall i \exists k, 1 \leq k \leq m, \) such that

\[
d_i(\sigma(w, \bar{y}^1, \ldots, \bar{y}^k)) \neq d_i(\sigma(\bar{w}, \bar{y}^1, \ldots, \bar{y}^k)),
\]

and

3. for some \( i \) and \( \bar{w} \in W_c \setminus \{ \bar{w} \} \), \( d_i(\bar{w}) \neq d_i(\bar{w}) \).

**Proof.** Given the outcome path \( h \in Y^\infty \) from the definition of separation, \( \sigma(w, h) \in W^\infty \) is the induced path of states with initial state \( w \in W \), and \( \sigma(w, h^t) \) is the state reached after the first \( t - 1 \) signals in \( h \). For each \( t, (\sigma(w, h^t))_{w \in R(\bar{w})} \) can be viewed as a vector in \( W^{R(\bar{w})} \). Since \( W \) is finite, there exists \( T_1 \) such that for all \( \tau \geq T_1 \), \( \sigma(\cdot, h^\tau) \) appears infinitely often in the sequence \( \{\sigma(\cdot, h^t)\}_t \). Let \( W^1 = \{\sigma(w, h^\tau) \) for some \( w \in R(\bar{w}) \) and \( \tau \geq T_1 \} \). There exists \( T'_1 \) such that for all \( w \in W^1, \sigma(w, T_1 h^{T_1}) = w \).

Consider now the sequence \( \{\sigma(w, T_1 h^{T_1+t})\}_{w \in R(\sigma(\bar{w}, T_1 h))} \). There exists \( T_2 \), such that for all \( \tau \geq T_1 + T_2 \), \( \sigma(\cdot, T_1 h^{T_1}) \) appears infinitely often in the sequence \( \{\sigma(\cdot, T_1 h^{T_1+t})\}_t \). Define \( W^2 = \{\sigma(w, T_1 h^{T_1}) \) for some \( w \in R(\sigma(\bar{w}, T_1 h)) \) and \( \tau \geq T_1 \} \). Clearly, \( W^1 \subset W^2 \), and there exists \( T'_2 \) such that for all \( w \in W^2, \sigma(w, T_2 h^{T_2}) = w \).
Since $W$ is finite, this process must eventually reach a point where $W_{k+1} = W_k$. We have thus identified a set of states $W^\kappa$ and two dates $T_k$ and $T'_k$, such that taking $T = T'_k$ as $(\bar{y}^1, \ldots, \bar{y}^m)$ and setting $\bar{w} = \sigma(\bar{w}, h^{T_k})$, parts 1 and 2 of the Lemma are satisfied.

If part 3 does not hold for this choice, by separation, it will hold in some period of the cycle $(\hat{y}^1, \ldots, \hat{y}^m)$, say period $\ell$. Part 3 then holds as well for the cycle beginning in period $\ell$, $(\bar{y}^\ell, \ldots, \bar{y}^m, \bar{y}^1, \ldots, \bar{y}^{\ell-1})$, and the state $\bar{w} = \sigma(\bar{w}, h^{T_k}, \bar{y}_1, \ldots, \bar{y}^{\ell-1})$.

It is easiest to first describe the behavior of beliefs of player $i$ over the private states of the other players under the limit private monitoring distribution $\pi^\infty$. Since $(\Omega, \pi^k)$ is strongly $1/k$-close to $(Y, \rho)$ and $\pi^k \to \pi^\infty$, for each $y \in Y$ the event $\{(\omega_1, \ldots, \omega_n) : \omega_i \in f_i^{-1}(y)\}$ is common belief under $\pi^\infty$. Moreover, if the other players start in the same state (such as $\bar{w}$) then they stay in the same state thereafter. We can thus focus on finding the appropriate sequence of signals to manipulate $i$’s updating about the current private states of the other players, without being concerned about the possibility that subsequent realizations will derail the process. We deal with that issue in a subsequent lemma. In the following lemma, a private signal $\omega_j$ for player $j$ is consistent with the private signal $\omega_i$ for player $i$ if $f_j(\omega_j) = f_i(\omega_i)$, where $f_i$ and $f_j$ are the signaling functions from Definition 4. It is an implication of this lemma that if player $i$ assigns strictly positive probability to all the other players being in the state $\bar{w}$, then after sufficient repetitions of the cycle $\bar{w}^L$, player $i$ eventually assigns probability arbitrarily close to 1 that at the end of a cycle, all the other players are in the state $\bar{w}$.

**Lemma 7** Suppose $(\Omega, \pi^\infty)$ is the limit of a rich sequence of private monitoring distributions. Fix a finite separating public profile of the public-monitoring game, and let $\bar{w}$, $\bar{w}^c$, and $i$ be the states, set of states, and player identified in Lemma 6. Then, there exists a finite sequence of private signals for player $i$, $\bar{w}^L_i \equiv (\omega^1_i, \omega^2_i, \ldots, \omega^L_i)$, such that

1. $\sigma_i(\bar{w}, \bar{w}^L_i) = \bar{w}$,
2. for all sequences of private signals, $\bar{w}^L_j$, for player $j \neq i$ consistent with $\bar{w}^L_i$, $\sigma_j(w, \bar{w}^L_j) = w$ for all $w \in W^c_i$, and
3. for all $w \in W^{n-1}_c \setminus \{\bar{w}1\}$,

\[
A(\bar{w}^L_i; w) \equiv \frac{\Pr^{\infty}(\bar{w}^L_i | w_{-i} = w, w_i = \bar{w})}{\Pr^{\infty}(\bar{w}^L_i | w_{-i} = \bar{w}1, w_i = \bar{w})} < 1,
\]

where $\Pr^{\infty}$ denotes probabilities calculated under $\pi^\infty$ and the assumption that all players follow the private profile at and after $t$.

**Proof.** The cycle $\hat{y}^1, \ldots, \hat{y}^m$ from Lemma 6 induces a cycle in the states $\bar{w} = \bar{w}^1, \ldots, \bar{w}^{m+1} = \hat{w}^1$ and $\bar{w} = \hat{w}^1, \ldots, \hat{w}^{m+1} = \hat{w}$. We index the cycle by $\ell$ and write
\( \bar{a}^\ell = d(\bar{w}^\ell) \) and \( \bar{a}_i^\ell = d_i(\bar{w}^\ell) \). Let \( \bar{a}^\ell \equiv (\bar{a}_i^\ell, \bar{a}_{-i}^\ell) \). Richness implies that for each \( \ell \), there exists a vector of nonnegative integers, \( (n_{\omega_i})_{\omega_i \in f_i^{-1}(y^\ell)} \), so that for all \( a_{-i}^\ell \neq \bar{a}_{-i}^\ell \),

\[
\sum_{\omega_i \in f_i^{-1}(y^\ell)} \gamma_{\bar{a}_i^\ell, a_{-i}^\ell} (\omega_i) n_{\omega_i} > 0.
\]

Since

\[
\gamma_{\bar{a}_i^\ell, a_{-i}^\ell} (\omega_i) = \log \left( \frac{\pi^\infty(\omega_i|\bar{a}^\ell)}{\pi^\infty(\omega_i|\bar{a}_i^\ell, a_{-i}^\ell)} \right)
\]

we have, for all \( a_{-i}^\ell \neq \bar{a}_{-i}^\ell \),

\[
\prod_{\omega_i \in f_i^{-1}(y^\ell)} \left( \frac{\pi^\infty(\omega_i|\bar{a}^\ell)}{\pi^\infty(\omega_i|\bar{a}_i^\ell, a_{-i}^\ell)} \right)^{n_{\omega_i}} > 1. \tag{4}
\]

Letting \( n_\ell = \sum_{\omega_i \in f_i^{-1}(y^\ell)} n_{\omega_i} \), for each \( \ell \), denote by \( N' \) the lowest common multiple of \( \{ n_1, \ldots, n_m \} \). Let \( \bar{c}_i^L \) denote the cycle of private signals for player \( i \) consistent with cycling \( N \) times through the public signals \( \bar{y}^1, \bar{y}^2, \ldots, \bar{y}^m \) and in which for each \( \ell \), the private signal \( \omega_i \in f_i^{-1}(y^\ell) \) appears \( (N'/n_\ell) n_{\omega_i} \) times. This cycle is of length \( L \equiv mN' \).

Given a private state profile \( w \in \mathbb{W}^{-1} \), let \( \bar{a}_{-i}^\ell \) denote the action profile taken in period \( \ell \) of the cycle. Then,

\[
A(\bar{c}_i^L; w) = \frac{\Pr_\infty(\bar{c}_i^L|w_{-i}^L = w, w_i = \bar{w})}{\Pr_\infty(\bar{c}_i^L|w_{-i}^L = \bar{w}1, w_i = \bar{w})} = \left( \prod_{\ell=1}^{m} \left( \prod_{\omega_i \in f_i^{-1}(y^\ell)} \left( \frac{\pi^\infty(\omega_i|\bar{a}_i^\ell, \bar{a}_{-i}^\ell)}{\pi^\infty(\omega_i|\bar{a}^\ell)} \right)^{n_{\omega_i}} \right)^{N'/n_\ell} \right).
\]

For \( w \neq \bar{w}1 \), then in each period at least one player is in a private state different from \( \bar{w} \). From Lemma 6.2, \( \bar{a}_{-i}^\ell \neq \bar{a}_{-i}^\ell \) for at least one \( \ell \), and so \( A(\bar{c}_i^L; w) \) must be strictly less than 1.

We are, of course, primarily concerned with private monitoring under the distribution \((\Omega, \pi^k)\). In this situation, one must deal with the possibility that player \( j \)'s private signals may be inconsistent with player \( i \)'s observations.

**Lemma 8** Assume the hypotheses of Lemma 7, and let \( h_i^k \) be a private history for player \( i \) satisfying \( \bar{w} = \sigma_i(h_i^k) \). For all \( \eta > 0 \), there exists \( \varepsilon > 0 \) and \( k' \) (independent of \( h_i^k \)) such that, for all \( k > k' \), if \( \eta < \Pr_k(w_{-i}^L \in \mathbb{W}^{-1} \setminus \{1\}|h_i^k) < 1 \) and \( \Pr_k(w_{-i}^L \notin \mathbb{W}^{-1} \setminus \{1\}|h_i^k) < \varepsilon \), then

\[
\frac{\Pr_k(w_{-i}^L \neq \bar{w}1|h_i^k, h_i^k)}{\Pr_k(w_{-i}^L = \bar{w}1|h_i^k, h_i^k)} < (1 - \varepsilon) \frac{\Pr_k(w_{-i}^L \neq \bar{w}1|h_i^k)}{\Pr_k(w_{-i}^L = \bar{w}1|h_i^k)}, \tag{5}
\]
where $\Pr_k$ denotes probabilities calculated under $\pi^k$ and the assumption that all players follow the private profile, and $\overline{\omega}_i^L$ is the sequence identified in Lemma 7.

**Proof.** For clarity, we suppress the conditioning on $h_i^t$. Denote the event that players other than $i$ observe some sequence of private signals consistent with the cycle $(\bar{y}_1^t, \ldots, \bar{y}_m^t)^N$ by $\bar{y}_{-i}$, and the complementary event by $\neg\bar{y}_{-i}$. Then,

$$
\Pr_k(w_{-i}^{t+L} \neq \bar{w} 1, \overline{\omega}_i^L) = \Pr_k(w_{-i}^{t+L} \neq \bar{w} 1, \overline{\omega}_i^L, \bar{y}_{-i}) + \Pr_k(w_{-i}^{t+L} \neq \bar{w} 1, \overline{\omega}_i^L, \neg\bar{y}_{-i})
$$

and

$$
\Pr_k(w_{-i}^{t+L} \neq \bar{w} 1, \overline{\omega}_i^L, \bar{y}_{-i}) \\
\leq \Pr_k(w_{-i}^{t} \neq \bar{w} 1, \overline{\omega}_i^L, \bar{y}_{-i}) \\
= \Pr_k(w_{-i}^{t} \in W_c^{n-1}\{\bar{w} 1\}, \overline{\omega}_i^L, \bar{y}_{-i}) + \Pr_k(w_{-i}^{t} \notin W_c^{n-1}\{\bar{w} 1\}, \overline{\omega}_i^L, \bar{y}_{-i})
$$

where the inequality arises because a player $j \neq i$ may be in a private state not in $W_c$. Now,

$$
\Pr_k(w_{-i}^{t} \in W_c^{n-1}\{\bar{w} 1\}, \overline{\omega}_i^L, \bar{y}_{-i}) \\
= \Pr_k(\overline{\omega}_i^L, \bar{y}_{-i}|w_{-i}^{t} \in W_c^{n-1}\{\bar{w} 1\}) \Pr_k(w_{-i}^{t} \in W_c^{n-1}\{\bar{w} 1\}) \\
\leq \Pr_k(\overline{\omega}_i^L, \bar{y}_{-i}|w_{-i}^{t} \in W_c^{n-1}\{\bar{w} 1\}) \Pr_k(w_{-i}^{t} \neq \bar{w} 1),
$$

and if $\Pr_k(w_{-i}^{t} \notin W_c^{n-1}\{\bar{w} 1\}) < \varepsilon$ (where $\varepsilon$ is to be determined),

$$
\Pr_k(w_{-i}^{t} \notin W_c^{n-1}\{\bar{w} 1\}, \overline{\omega}_i^L, \bar{y}_{-i}) + \Pr_k(w_{-i}^{t+L} \neq \bar{w} 1, \overline{\omega}_i^L, \neg\bar{y}_{-i}) \\
< \varepsilon + \Pr_k(w_{-i}^{t+L} \neq \bar{w} 1, \overline{\omega}_i^L, \neg\bar{y}_{-i}) \\
\leq \varepsilon + \Pr_k(\overline{\omega}_i^L, \neg\bar{y}_{-i}) \\
= \varepsilon + \Pr_k(\neg\bar{y}_{-i} | \overline{\omega}_i^L) \Pr_k(\overline{\omega}_i^L).
$$

Moreover,

$$
\Pr_k(w_{-i}^{t+L} = \bar{w} 1, \overline{\omega}_i^L) \geq \Pr_k(w_{-i}^{t} = \bar{w} 1, \overline{\omega}_i^L, \bar{y}_{-i}) \\
= \Pr_k(\overline{\omega}_i^L, \bar{y}_{-i}|w_{-i}^{t} = \bar{w} 1) \Pr_k(w_{-i}^{t} = \bar{w} 1).
$$

Defining

$$
a^t(k) \equiv \frac{1}{\Pr_k(w_{-i}^{t} \neq \bar{w} 1)} \left( \varepsilon + \Pr_k(\neg\bar{y}_{-i} | \overline{\omega}_i^L) \Pr_k(\overline{\omega}_i^L) \right),
$$
we have,
\[
\frac{\Pr_k(w^{t+L}_{-i} \neq \bar{w}1 | \omega^L_i)}{\Pr_k(w^{t+L}_{-i} = \bar{w}1 | \omega^L_i)} < \frac{\Pr_k(\omega^L_i, \bar{g}_{-i}|w^t_{-i} \in W_{c^{\omega^L_i-1}} \{\bar{w}1\}) + x^t(k)}{\Pr_k(\omega^L_i, \bar{g}_{-i}|w^t_{-i} = \bar{w}1)} \times \frac{\Pr_k(w^{t+L}_{-i} \neq \bar{w}1)}{\Pr_k(w^{t+L}_{-i} = \bar{w}1)}
\]
\[
\leq \max_{w \in W_{c^{\omega^L_i-1}} \{\bar{w}1\}} \frac{\Pr_k(\omega^L_i, \bar{g}_{-i}|w^t_{-i} = w) + x^t(k)}{\Pr_k(\omega^L_i, \bar{g}_{-i}|w^t_{-i} = \bar{w}1)} \times \frac{\Pr_k(w^{t+L}_{-i} \neq \bar{w}1)}{\Pr_k(w^{t+L}_{-i} = \bar{w}1)}.
\]
(6)

From Lemma 7,
\[
\max_{w \in W_{c^{\omega^L_i-1}} \{\bar{w}1\}} A(\omega^L_i; w) = \max_{w \in W_{c^{\omega^L_i-1}} \{\bar{w}1\}} \lim_{k \to \infty} \frac{\Pr_k(\omega^L_i, \bar{g}_{-i}|w^t_{-i} = w)}{\Pr_k(\omega^L_i, \bar{g}_{-i}|w^t_{-i} = \bar{w}1)} < 1,
\]
and so there is an there is an $\varepsilon' > 0$ sufficiently small so that (recall that the denominator has a strictly positive limit)
\[
\max_{w \in W_{c^{\omega^L_i-1}} \{\bar{w}1\}} \lim_{k \to \infty} \frac{\Pr_k(\omega^L_i, \bar{g}_{-i}|w^t_{-i} = w) + \varepsilon'}{\Pr_k(\omega^L_i, \bar{g}_{-i}|w^t_{-i} = \bar{w}1)} < 1 - \varepsilon'.
\]
The finiteness of the state space and the number of players allows us to interchange the max and limit operations. Consequently, there exists $k''$ such that for all $k \geq k''$,
\[
\max_{w \in W_{c^{\omega^L_i-1}} \{\bar{w}1\}} \frac{\Pr_k(\omega^L_i, \bar{g}_{-i}|w^t_{-i} = w) + \varepsilon'}{\Pr_k(\omega^L_i, \bar{g}_{-i}|w^t_{-i} = \bar{w}1)} < 1 - \varepsilon'.
\]
(7)

Since $(\Omega, \pi^k)$ is strongly $1/k$-close to $(Y, \rho)$, $\lim_{k \to \infty} \Pr_k(\neg \bar{g}_{-i}|\omega^L_i) = 0$, and so there exists $k''$ such that $\Pr_k(\neg \bar{g}_{-i}|\omega^L_i) < \varepsilon' \eta/2$ for all $k \geq k''$. Suppose $\varepsilon = \varepsilon' \eta/2$ and $k' = \max\{k'', k''\}$. Since $\eta < \Pr_k(w^t_{-i} \in W_{c^{\omega^L_i-1}} \{\bar{w}1\}) \leq \Pr_k(w^t_{-i} \neq \bar{w}1)$, $x^t(k) \leq \varepsilon'$. Consequently (7), with (6), implies (5) (since $\varepsilon < \varepsilon'$).

Lemma 9 Assume the hypotheses of Lemma 7, and let $h^t_i$ be a private history for player $i$ satisfying $\dot{w} = \sigma_i(h^t_i)$. Fix $\eta > 0$ and let $\varepsilon$ and $k'$ be the constants identified in Lemma 8. There exists $T$ such that if $t \geq T$, then for all $k > k'$,
\[
\Pr_k(w^{t+L}_{-i} \notin W_{c^{\omega^L_i-1}} | \omega^L_i, h^t_i) < \varepsilon.
\]
**Proof.** Fix $T$ large enough, so that if $\hat{w} \in W_t$ (the set of states reachable in period $t$) for $t \geq T$, then $W_t \subset R(\hat{w})$. Separation then implies $\Pr_k(w_{-i}^{t+L} \notin \tilde{W}'_{c_{i}} \setminus \hat{w}) = 0$, and so

\[
\Pr_k(\omega_{c_{i}}^{t+L} \notin \tilde{W}'_{c_{i}} | \omega_{i}^{L})
= \Pr_k(\omega_{-i}^{t+L} \notin \tilde{W}'_{c_{i}} \setminus \hat{w}) + \Pr_k(\omega_{-i}^{t+L} \notin \tilde{W}'_{c_{i}} \setminus \hat{w})
= \Pr_k(\omega_{-i}^{t+L} \notin \tilde{W}'_{c_{i}} \setminus \hat{w})
\leq \Pr_k(\omega_{-i}^{t+L} | \omega_{i}^{L}),
\]

which is less than $\varepsilon$ for $k \geq k'$. \(\blacksquare\)

We are now in a position to complete the proof. Suppose $\hat{h}_i$ is a private history for player $i$ that leads to the private state $\hat{w}$ with $t > T$, and let $\eta$ be the constant implied by Theorem 3. Since $\hat{w}$ and $\hat{w}$ are both reachable in the same period, with positive probability player $i$ observes a private history $\hat{h}_i$ that leads to the private state $\hat{w}$. Moreover, at $\hat{h}_i$ his posterior beliefs that all the other players are in the private state $\hat{w}$, $\Pr_k(w_{-i}^{t-1} = \hat{w})$, is strictly positive for all $k$ (where $\Pr_k$ denotes probabilities under $\pi_k$). If $\Pr_k(w_{-i}^{t-1} = \hat{w}) > \eta$, then $\Pr_k(w_{-i}^{t-1} = \hat{w}) < 1 - \eta$, and since $d_i(\hat{w}) = d_i(\hat{w})$, Theorem 3 yields the desired conclusion.

Suppose then that $\Pr_k(w_{-i}^{t-1} = \hat{w}) > \eta$, and $k > k'$, where $k'$ is from Lemma 8. Lemmas 8 and 9 immediately imply that, as long as $\Pr_k(w_{-i}^{t+\kappa} = \hat{w}) > \eta$, after the first cycle, the odds ratio falls off until eventually, $\Pr_k(w_{-i}^{t+\kappa} = \hat{w}) \leq \eta$, at which point we are in the first case (since $\hat{w}$ cycles under $\omega_{i}^{L}$, $i$'s private state continually returns to $\hat{w}$).

**A. Omitted Proofs**

**Proof of Lemma 1.** For any sequence of private signals $(\omega_i^1, \omega_i^2, \ldots)$, the history of actions induced by $s_i$ is given recursively by

- $\bar{a}_i^1 = s_i(\emptyset)$,
- $\bar{a}_i^2 = s_i(\bar{a}_i^1, \omega_i^1)$,
- $\bar{a}_i^3 = s_i(\bar{a}_i^1, \omega_i^1, \bar{a}_i^2, \omega_i^2)$,
- $\vdots$

The action-free strategy $\tilde{s}_i$ that is realisation equivalent to $s_i$ is then given by

\[
\tilde{s}_i(\bar{a}_i^1, \omega_i^1, \bar{a}_i^2, \omega_i^2, \ldots, \bar{a}_i^{t-1}, \omega_i^{t-1}) = s_i(\bar{a}_i^1, \omega_i^1, \bar{a}_i^2, \omega_i^2, \ldots, \bar{a}_i^{t-1}, \omega_i^{t-1}).
\]

\(\text{For fixed } \hat{h}_i, \Pr_k(w_{-i} \notin \hat{w}) \to 0 \text{ as } k \to \infty.\)
It is now immediate that every mixed strategy is realization equivalent to a mixture over action-free strategies. 

Proof of Lemma 4. Suppose \((u^*, (\Omega, \pi))\) is \(\varepsilon\)-close to \((\tilde{u}^*, (Y, \rho))\) with associated signaling functions \((f_1, \ldots, f_n)\). Then, for all \(a\),

\[
\left| \sum_{\omega_1, \ldots, \omega_n} u_i^*(\omega_i, a_i)\pi(\omega_1, \ldots, \omega_n|a) - \sum_{y_i} \tilde{u}_i^*(y_i, a_i)\rho(y|a) \right| \\
\leq \left| \sum_y \left( \sum_{\omega_1 \in f_1^{-1}(y)} \ldots \sum_{\omega_n \in f_n^{-1}(y)} u_i^*(\omega_i, a_i)\pi(\omega_1, \ldots, \omega_n|a) - \tilde{u}_i^*(\omega_i, a_i)\rho(y|a) \right) \right| \\
+ |Y| \varepsilon \max_{\omega_i, a_i} |u_i^*(\omega_i, a_i)| \\
\leq 2 |Y| \varepsilon \max_{\omega_i, a_i} |u_i^*(\omega_i, a_i)| + \varepsilon + \varepsilon^2 |Y|,
\]

where the first inequality follows from \(\sum_y \pi(\{\omega : f_i(\omega_i) = y\} | a) > 1 - \varepsilon |Y|\) (an implication of part 1 of Definition 4), the second equality follows from \(|\tilde{u}_i^*(y, a_i) - u_i^*(\omega_i, a_i)| < \varepsilon\) for all \(i \in N, a_i \in A_i\), and \(\omega_i \in f_i^{-1}(y)\), and the third inequality from part 1 of Definition 4 and \(\max_{y, a_i} |\tilde{u}_i^*(y, a_i)| \leq \max_{\omega_i, a_i} |u_i^*(\omega_i, a_i)| + \varepsilon\). The last term can clearly be made smaller than \(\eta\) by appropriate choice of \(\varepsilon\).

Proof of Lemma 5. Suppose there exists \(L\) such that for all \(w, w' \in W\) reachable in the same period and for all \(h \in Y^\infty\),

\[\sigma(w, h^L) = \sigma(w', h^L).\]

Then, for all \(w, w' \in W\) reachable in the same period and for all \(h \in Y^\infty\),

\[d(\sigma(w, h^t)) = d(\sigma(w', h^t)) \quad \forall t \geq L + 1.\]

If \(w = \sigma(w^1, y^1, \ldots, y^{t-L-1})\) and \(w' = \sigma(w^1, \hat{y}^1, \ldots, \hat{y}^{t-L-1})\), then

\[s(h^1) = d(\sigma(w, y^{t-L}, \ldots, y^{t-1})) = d(\sigma(w', y^{t-L}, \ldots, y^{t-1})) = d(\sigma(w', \hat{y}^{t-L}, \ldots, \hat{y}^{t-1})) = s(\hat{h}^t).\]
Suppose now the profile $s$ has bounded recall. Let $(W, w^1, \sigma, d)$ be a representation of $s$. Suppose $w$ and $w'$ are two states reachable in the same period. Then there exists $h^\tau$ and $h^\tau$ such that $w = \sigma(w^1, h^\tau)$ and $w' = \sigma(w^1, h^\tau)$. Then, for all $h \in Y^\infty$, $(h^\tau, h'^\tau)$ and $(h^\tau, h'^\tau)$ agree for the last $t - 1$ periods, and so if $t \geq L + 1$, they agree for at least the last $L$ periods, and so

$$d(\sigma(w, h')) = s(h^\tau, h')$$

$$= s(h^\tau, h') = d(\sigma(w', h'^\tau)).$$

Minimality of the representing automaton then implies that for all $h \in Y^\infty$ and $w, w' \in W$ reachable in the same period, $\sigma(w, h^\tau) = \sigma(w', h'^\tau)$.

**Proof of Theorem 3.** Let $\phi_i(w)$ be player $i$’s continuation value from the strategy profile $(W, w, \sigma, d)$ in the game with public monitoring (i.e., $\phi_i(w)$ is the continuation value of state $w$ under the profile $(W, w^1, \sigma, d)$), and let $\phi_i(s_i|w)$ be the continuation value to player $i$ from following the strategy $s_i$ when all the other players follow the strategy profile $(W, w, \sigma, d)$. Since the public profile is a strict equilibrium and $|W| < \infty$, there exists $\theta > 0$ such that for all $i$, $w \in W$, and $\tilde{s}_i$, a deviation continuation strategy for player $i$ with $\tilde{s}_i \neq d_i(w)$,

$$\phi_i(\tilde{s}_i|w) < \phi_i(w) - \theta.$$

Every strategy $\tilde{s}_i$ in the game with public monitoring induces a strategy $s_i$ in the games with private monitoring that are strongly $\varepsilon$-close in the natural manner:

$$s_i(a^1_i, \omega^1_i; a^2_i, \omega^2_i; \ldots, a^{t-1}_i, \omega^{t-1}_i) = \tilde{s}_i(a^1_i, f_i(\omega^1_i); a^2_i, f_i(\omega^2_i); \ldots, a^{t-1}_i, f_i(\omega^{t-1}_i)).$$

Denote by $V^\pi_i(w)$ the expected value to player $i$ in the game with private monitoring $(w^*, (\Omega, \pi))$ from the private profile induced by $(W, w, \sigma, d)$. Let $V^\pi_i(s_i|h^\pi_i)$ denote player $i$’s continuation value of a strategy $s_i$ in the game with private monitoring, conditional on the private history $h^\pi_i$.

There exists $\varepsilon$ and $\eta > 0$ such that for all strategies $\tilde{s}_i$ for player $i$ in the game with public monitoring, and all histories $h^\tau_i$ for $i$ in the game with private monitoring, if the game with private monitoring is strongly $\varepsilon$-close to the game with public monitoring and $\beta_i(w^1|h^\pi_i) > 1 - \eta$, then $|V^\pi_i(s_i|h^\pi_i) - \phi_i(\tilde{s}_i|w)| < \theta/3$, where $s_i$ is the induced strategy in the game with private monitoring. (The argument is essentially the same as that of Mailath and Morris (2002, Lemma 3).)

Suppose there exists a player $i$, a private history $h^\pi_i$, and a state $w$ such that $d_i(w) \neq d_i(\sigma_i(h^\pi_i))$ and $\beta_i(w^1|h^\pi_i) > 1 - \eta$. Denote by $s'_i$ the private strategy described by $(W, w, \sigma_i, d_i)$, $\tilde{s}'_i$ the public strategy described by $(W, w, \sigma, d_i)$, $s_i$ the private strategy
described by \((W, \sigma_i(h^t_i), \sigma, d_i)\), and \(\tilde{s}_i\) the public strategy described by \((W, \sigma_i(h^t_i), \sigma, d_i)\). Then,

\[
V^\pi_i(s'_i|h^t_i) > \phi_i(\tilde{s}'_i|w) - \theta/3 = \phi_i(w) - \theta/3 \\
> \phi_i(\tilde{s}_i|w) + 2\theta/3 \\
> V^\pi_i(s_i|h^t_i) + \theta/3 \\
= V^\pi_i(\sigma_i(h^t_i)) + \theta/3,
\]

so that \(s'_i\) is a profitable deviation.

References


